which gives for a new value of i, $\cdot 11 + \cdot 00221 = \cdot 11221$. Trying this we get for result 27·1983, which is quite near enough, the error being only $= \cdot 0088$. I have computed a few cases, and consigned the results to the small table below; but it is unnecessary to give further details. The annual rate is found from the half-yearly rate by the formula, $(1+i)^2 - 1$.

n.	Half-yearly Rate per Cent.	Error.	Annual Rate per Cent.
5	11·221	+·0088	23·700
25	4·708	+·0008	9·635
45	4·062	−·0022	8·288
65	3·849	−·0019	7·845
78	3·783	−·0055	7·709
Limit	3·678		7·490

On the Theory of Annuities Certain. *By* WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

THE interest with which I have perused Mr. Gray's very able paper in the last number of the *Journal*, has led me to give some consideration to the general theory of varying Annuities; and as the results of the investigation appear to me somewhat striking, I am induced to think that a brief paper on the subject may not be unacceptable to that section of the scientific world to which this *Journal* more particularly addresses itself.

Nearly all that has hitherto appeared in connection with this subject is comprised, I believe, in Chapters XV. and XVI. of Baily's work on Interest and Annuities.* Like everything which proceeded from the hand of that excellent writer, the matter is treated with skill and perspicuity; but owing to the prevailing want of appreciation of the importance of the inquiry (an importance which Mr. Gray has been the first to demonstrate), and the restricted scope of the object which the author had in consequence proposed to himself, the result is very far indeed from satisfactory. In proof of this assertion I need only mention the fact that

* This remark is applicable only to Annuities Certain, as I find that Mr. Gray himself, in his work on Life Contingencies, has treated upon Life Annuities the successive payments of which are the several orders of figurate numbers. The connection between the values of Annuities of two successive orders is shewn by Mr. Gray to be expressed by the equation $\dot{i}_x^{(m)} = \dot{i}_x^{(m-1)} + vp_x \dot{i}_{x+1}^{(m)}$, where $\dot{i}_x^{(m)}$ denotes the value of an annuity of the mth order of figurate numbers, on a life aged x. In Mr. G.'s arrangement the payment due at the end of the first year is always unity.

Mr. Gray, in investigating his problem, has not attempted to avail himself of Baily's researches, but has applied a very general and useful formula which he had himself brought forward in a former paper devoted to a totally different subject. Having a specific object in view, viz., the numerical solution of a particular problem which had been erroneously treated by another writer, Mr. Gray has naturally used his formula in its readiest form, and has not gone out of his way to trace its connection with the general system by which the doctrine of Annuities is usually expounded. If therefore it shall appear that by following a different course a more convenient expression might have been obtained, such a circumstance in no wise detracts from the merits of Mr. Gray's performance, seeing that the labour of the necessary preliminary investigation would have greatly outweighed any saving which might thereby have been effected in the numerical solution of the problem.

LEMMA. (I.)

Let an annuity the xth payment of which is unity, be termed an annuity of the first order; an annuity, the xth payment of which is $(x-1)$, an annuity of the second order; and, generally, an annuity the xth payment of which is $\frac{(x-1)(x-2)\ldots(x-t+1)}{1.2\ldots.(t-1)}$, an annuity of the tth order. Also let $(1+i)^n$ be represented by $\overset{0}{A}_n$, and let $\overset{1}{A}_n$ denote the amount of an annuity (for n years) of the first order, $\overset{2}{A}_n$ the amount of an annuity (for the like term) of the second order, and so on. Further let $[\overset{t}{x}]$ denote the xth payment of an annuity of the tth order. Then I say that the law of the series $\overset{0}{A}_n$, $\overset{1}{A}_n$, $\overset{2}{A}_n$.... shall be expressed by the equation

$$\overset{t}{A}_n = \frac{\overset{t-1}{A}_n - [\overset{t}{n+1}]}{i}$$

Demonstration.

When $t=1$, the equation becomes $\overset{1}{A}_n = \frac{\overset{0}{A}_n - [\overset{1}{n+1}]}{i} = \frac{(1+i)^n - 1}{i}$, the truth of which is proved in all works on the doctrine of Annuities Certain. The following Table shewing the successive payments of Annuities of different orders will be useful in dealing with higher values of t.

x.	1st Order.	2nd Order.	3rd Order.	4th Order.	5th Order.	6th Order.	7th Order.	8th Order.
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0
4	1	3	3	1	0	0	0	0
5	1	4	6	4	1	0	0	0
6	1	5	10	10	5	1	0	0
7	1	6	15	20	15	6	1	0
8	1	7	21	35	35	21	7	1
&c.	1	&c.	&c.	&c.	&c.	&c.	&c.	&c.

By a well known property of these series we have

$$\overset{t}{[n]}=\overset{t}{[n-1]}+\overset{t-1}{[n-1]}.$$

Now, $\overset{t}{A}_n=\overset{t}{[n]}+\overset{t}{[n-1]}(1+i)+\overset{t}{[n-2]}(1+i)^2+\ldots+\overset{t}{[1]}(1+i)^n$.

And $\overset{t-1}{A}_n=\overset{t-1}{[n]}+\overset{t-1}{[n-1]}(1+i)+\overset{t-1}{[n-2]}(1+i)^2+\ldots+\overset{t-1}{[1]}(1+i)^n$.

Hence, by addition,

$$\overset{t}{A}_n+\overset{t-1}{A}_n=\overset{t}{[n+1]}+\overset{t}{[n]}(1+i)+\overset{t}{[n-1]}(1+i)^2+\ldots+\overset{t}{[2]}(1+i)^n=\overset{t}{A}_{n+1},$$

as for all values of t greater than 1, $\overset{t}{[1]}$ vanishes. Therefore

$$\overset{t}{A}_n+\overset{t-1}{A}_n=\overset{t}{A}_{n+1}.$$

But $$\overset{t}{A}_{n+1}=\overset{t}{A}_n(1+i)+\overset{t}{[n+1]}.$$

Therefore $$\overset{t-1}{A}_n=\overset{t}{A}_n i+\overset{t}{[n+1]};$$

and finally, $$\overset{t}{A}_n=\frac{\overset{t-1}{A}_n-\overset{t}{[n+1]}}{i}$$

Cor. Putting t successively equal to 1, 2, 3, 4, &c., we have

$$\overset{1}{A}_n=\frac{(1+i)^n-1}{i}$$

$$\overset{2}{A}_n=\frac{\overset{1}{A}_n-n}{i}$$

$$\overset{3}{A}_n=\frac{\overset{2}{A}_n-n\dfrac{n-1}{2}}{i}$$

$$\overset{4}{A}_n=\frac{\overset{3}{A}_n-n\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}}{i}$$

&c. = &c.

As an example of the use of this theorem let it be required to find the successive values of $\overset{2}{A}_5$, $\overset{3}{A}_5 \ldots \overset{5}{A}_5$, when $i = {\cdot}05$.

$n=5$ $\qquad \overset{1}{A}_5 = 5{\cdot}52563125$ (Baily, *Int. & Anns.*, Table V.)

$\qquad\qquad {\cdot}52563125 \times 20$

$n\dfrac{n-1}{2}=10 \qquad \overset{2}{A}_5 = 10{\cdot}5126250$

$\qquad\qquad {\cdot}5126250 \times 20$

$n\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}=10 \qquad \overset{3}{A}_5 = 10{\cdot}252500$

$\qquad\qquad {\cdot}252500 \times 20$

$n\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}\cdot\dfrac{n-3}{4}=5 \qquad \overset{4}{A}_5 = 5{\cdot}05000$

$\qquad\qquad {\cdot}05000 \times 20$

$\overset{5}{A}_5 = 1{\cdot}0000$

The final result proves the accuracy of the process, as $\overset{n}{A}_n$ is always unity.

LEMMA. (II.)

Let unity be denoted by $\overset{0}{V}_n$ and let $\overset{1}{V}_n$ denote the value of an Annuity of the first order, $\overset{2}{V}_n$ the value of an annuity of the second order, and so on. The law of the series $\overset{0}{V}_n$, $\overset{1}{V}_n$, $\overset{2}{V}_n \ldots$ shall be expressed by the equation :

$$\overset{t}{V}_n = \frac{\overset{t-1}{V}_n - [\overset{t}{n+1}](1+i)^{-n}}{i}$$

Demonstration.

$$\overset{t}{V}_n = [\overset{t}{1}](1+i)^{-1} + [\overset{t}{2}](1+i)^{-2} + \ldots + [\overset{t}{n}](1+i)^{-n}$$

$$\overset{t-1}{V}_n = [\overset{t-1}{1}](1+i)^{-1} + [\overset{t-1}{2}](1+i)^{-2} + \ldots + [\overset{t-1}{n}](1+i)^{-n}$$

$$\therefore \overset{t}{V}_n + \overset{t-1}{V}_n = [\overset{t}{2}](1+i)^{-1} + [\overset{t}{3}](1+i)^{-2} + \ldots + [\overset{t}{n+1}](1+i)^{-n} = \overset{t}{V}_{n+1}(1+i).$$

But $\overset{t}{V}_{n+1} = \overset{t}{V}_n + [\overset{t}{n+1}](1+i)^{-\overline{n+1}}$. Hence $\overset{t}{V}_n + \overset{t-1}{V}_n = \overset{t}{V}_n(1+i) + [\overset{t}{n+1}](1+i)^{-n}$ and $\overset{t}{V}_n i = \overset{t-1}{V}_n - [\overset{t}{n+1}](1+i)^{-n}$. Finally we have

$$\overset{t}{V}_n = \frac{\overset{t-1}{V}_n - [\overset{t}{n+1}](1+i)^{-n}}{i}$$

The case in which $t=1$ is proved in works on Annuities Certain,—the equation then becoming $\overset{1}{V}_n = \dfrac{1-(1+i)^{-n}}{i}$.

Cor. Putting t successively equal to 1, 2, 3, 4, &c. we have

$$\overset{1}{V}_n = \frac{1-(1+i)^{-n}}{i}$$

$$\overset{2}{V}_n = \frac{\overset{1}{V}_n - n(1+i)^{-n}}{i}$$

$$\overset{3}{V}_n = \frac{\overset{2}{V}_n - n\frac{n-1}{2}(1+i)^{-n}}{i}$$

$$\overset{4}{V}_n = \frac{\overset{3}{V}_n - n\frac{n-1}{2}\cdot\frac{n-2}{3}(1+i)^{-n}}{i}$$

&c. = &c.

In illustration of this Theorem I propose to determine successively the values of $\overset{1}{V}_{40}$, $\overset{2}{V}_{40}$, and $\overset{3}{V}_{40}$ when interest is taken at 5 per cent.

$$(1+i)^{-40} = \cdot 14204568$$

$$40(1+i)^{-40} = 5\cdot 6818272$$

$$\begin{array}{r} 113\cdot 636544 \\ 2\cdot 840914 \\ \hline \end{array}$$

$$40 \times \frac{39}{2}(1+i)^{-40} = 110\cdot 795630$$

$$\begin{array}{rr}
 & 1\cdot \\
 & \cdot 1420457 \\ \hline
 & \cdot 8579543 \times 20 \\ \hline
\overset{1}{V}_{40} = & 17\cdot 159086 \\
 & 5\cdot 681827 \\ \hline
 & 11\cdot 477259 \times 20 \\ \hline
\overset{2}{V}_{40} = & 229\cdot 54518 \\
 & 110\cdot 79563 \\ \hline
 & 118\cdot 74955 \times 20 \\ \hline
\overset{3}{V}_{40} = & 2374\cdot 9910
\end{array}$$

PROBLEM. (I.)

To determine the amount of an annuity for n years, the successive payments of which are $u_1, u_2, u_3 \ldots u_n$.

Solution.

Substituting for u_2, u_3, &c. their respective values in terms of u_1, Δu_1, $\Delta^2 u_1$, &c. we have

$$u_1 = u_1$$
$$u_2 = u_1 + \Delta u_1$$
$$u_3 = u_1 + 2\Delta u_1 + \Delta^2 u_1$$
$$u_4 = u_1 + 3\Delta u_1 + 3\Delta^2 u_1 + \Delta^3 u_1$$
$$\cdot \quad \cdot$$
$$u_n = u_1 + (n-1)\Delta u_1 + (n-1)\frac{n-2}{2}\Delta^2 u_1 + \ldots . + \Delta^{n-1} u_1$$

Hence it appears that the given annuity is equivalent to an annuity of the first order, for n years, of u_1, + an annuity of the second order, for the same term, of Δu_1, + an annuity of the third order, for the same term, of $\Delta^2 u_1$, and so on until the last significant order of differences. The required amount is therefore

$$\overset{1}{A}_n . u_1 + \overset{2}{A}_n . \Delta u_1 + \overset{3}{A}_n . \Delta^2 u_1 + \ldots .$$

Example.—Required the amount of an annuity for 5 years at 5 per cent the successive payments of which are 5^3, 4^3, 3^3, 2^3, 1^3.

Here we have

$$\begin{array}{lllll} u_1 = 125 & & & & \\ & -61 & & & \\ u_2 = 64 & & +24 & & \\ & -37 & & -6 & \\ u_3 = 27 & & +18 & & \\ & -19 & & -6 & \\ u_4 = 8 & & +12 & & \\ & -7 & & & \\ u_5 = 1 & & & & \end{array}$$

and therefore the amount of the given annuity is

$$125\overset{1}{A}_5 - 61\overset{2}{A}_5 + 24\overset{3}{A}_5 - 6\overset{4}{A}_5.$$

The example in the case of Theorem I. gives the values of $\overset{1}{A}_5$, &c.

5·52563125 × 125 − 10·512625 × 61 + 10·2525000 × 24 − 5·050000 × 6

1·38140781	630·757500		
		205·050000	30·300000
690·703906	641·270125	41·010000	
246·060000	30·300000		
		246·060000	
936·763906	671·570125		
671·570125			
265·193781 (Answer.)			

When there are three orders of differences and only five terms little is gained by the use of the formula, as each term of the annuity may easily be valued separately. I have however selected this example because it is one given by Baily, who gets for the answer 265·19378125,—showing that mine is correct to the last decimal place. Baily probably selected this case for the facility of

testing his result; a very proper precaution for an author to observe on breaking new ground.

I was surprised to observe that Baily designates the above an *Increasing Annuity*,—by which term (when treating of the *amounts* of annuities) he invariably understands an annuity whose last payment is unity and whose preceding payments are greater than unity.*

PROBLEM. (II.)

To determine the present value of an annuity (for n years) the successive payments of which are $u_1, u_2, u_3 \ldots u_n$.

Solution.

Substituting for u_2, u_3, &c. their respective values in terms of u_1, Δu_1, &c., as in the solution of the preceding Problem, and taking the present value of each order, we have for the value required

$$\overset{1}{V}_n . u_1 + \overset{2}{V}_n . \Delta u_1 + \overset{3}{V}_n . \Delta^2 u_1 + \ldots .$$

Example.—Required the value of an annuity for 40 years at 5 per cent. the successive payments of which are 417500, 424680, 431620, &c. the third differences of the series being 0.

Differencing we have

$$\begin{array}{lll} u_1 = 417500 & & \\ & +7180 & \\ u_2 = 424680 & & -240 \\ & +6940 & \\ u_3 = 431620 & & \end{array}$$

Hence the required present value is $417500 \times \overset{1}{V}_{40} + 7180 \times \overset{2}{V}_{40} - 240 \times \overset{3}{V}_{40}$; and taking the values of $\overset{1}{V}_{40}$, &c., from the example given in the second Theorem we have

$$\begin{array}{lll} \log \overset{1}{V}_{40} = 1{\cdot}2344943 & \log \overset{2}{V}_{40} = 2{\cdot}3608683 & \log \overset{3}{V}_{40} = 3{\cdot}3756620 \\ \log u_1 = 5{\cdot}6206565 & \log \Delta u_1 = 3{\cdot}8561244 & \log \Delta^2 u_1 = 2{\cdot}3802112 \\ \hline \quad 6{\cdot}8551508 & \quad 6{\cdot}2169927 & \quad 5{\cdot}7558732 \end{array}$$

* Since writing the above Mr. Adler has shewn me a presentation copy of Baily's work, which belonged to the late Mr. Gompertz, in which the whole of the investigation relative to the *amounts* of increasing annuities is cancelled and the following note, in Baily's own handwriting, appended: "The whole of this investigation is erroneous "*in principle*,—it was written in haste whilst the proof was waiting. (Signed) F. B." This explains the circumstance alluded to in the text.

Mr. Adler has also pointed out to me that the materials of Baily's two chapters are to be found in the second volume of Dodson's *Mathematical Repository*, to which work, indeed, Baily refers his readers for further information on the subject.

$$\begin{array}{r} +7{,}163{,}922 \\ +1{,}648{,}135 \\ \hline 8{,}812{,}057 \\ -\ \ 569{,}998 \\ \hline 8{,}242{,}059 \end{array} \quad \text{(Answer.)}$$

This is one of the values determined by Mr. Gray in the solution of his problem.

If it be required to determine the present value of a perpetuity the successive payments of which are $u_1\ u_2\ u_3 \ldots$ (a series of which the differences are supposed to vanish after a definite number of orders) we are led into a discussion which demands the application of the higher mathematics. In the case of Annuities of the first order, no difficulty arises; for $\overset{1}{V}_n$ being $\frac{1-(1+i)^{-n}}{i}$, the effect of making n infinitely great is evident by inspection, as the term $(1+i)^{-n}$ vanishes, and we have $\overset{1}{V}_\infty = \frac{1}{i}$, as is shown in all works on Annuities Certain. But when we come to the next order, $\overset{2}{V}_n$, which is equal to $\frac{\overset{1}{V}_n - n(1+i)^{-n}}{i}$, where the second term consists of two factors one of which *increases*, while the other *decreases*, and both without limit, the ultimate result is by no means so apparent. A simple application of the Differential Calculus, however, shows that the term in question vanishes, like the corresponding term in the expression for $\overset{1}{V}_n$. When the numerator and denominator of a fraction both increase without limit, to determine the limit of the value of the fraction, we must substitute for the numerator and denominator their respective differential coefficients, repeating the process, if necessary, until a fraction is obtained of a form exhibiting its ultimate or limiting value. In the case of $n(1+i)^{-n}$, or $\frac{n}{(1+i)^n}$, a single substitution gives $\frac{1}{\log(1+i) \times (1+i)^n}$, which evidently vanishes when n is increased without limit; and hence we conclude that the primitive fraction, $\frac{n}{(1+i)^n}$, does the same. We have then $\overset{2}{V}_\infty = \frac{\overset{1}{V}_\infty}{i} = \frac{1}{i^2}$.

In the case of Annuities of the third order we have $\overset{3}{V}_n = \dfrac{\overset{2}{V}_n - n\dfrac{n-1}{2}(1+i)^{-n}}{i}$. Making the requisite substitution in the fraction $\dfrac{\frac{1}{2}n(n-1)}{(1+i)^n}$, we obtain $\dfrac{n-\frac{1}{2}}{\log(1+i).(1+i)^n}$, which however does not answer our purpose, as the numerator and denominator both increase without limit with n. But repeating the process upon the expression last obtained we get $\dfrac{1}{\{\log(1+i)\}^2(1+i)^n}$, which enables us to see by inspection that the limit we are in search of is 0, as in the two preceding cases. We have therefore $\overset{3}{V}_\infty = \dfrac{\overset{2}{V}_\infty}{i} = \dfrac{1}{i^3}$.

Proceeding in the same way with the higher orders of Annuities, we shall find that to determine the limit of the term $\dfrac{n(n-1)..(n-t+2)}{1.2\,....\,t-1}(1+i)^{-n}$,—which is the $(n+1)$th payment of the tth order,—we should have to perform $t-1$ substitutions, obtaining for the final result $\dfrac{1}{\{\log(1+i)\}^{t-1}(1+i)^n}$. Hence it appears that the law holds good for any finite number of orders, as the expression last given vanishes when n *does,* and t *does not,* increase without limit; and we have therefore for the value of the perpetuity:

$$\frac{u_1}{i} + \frac{\Delta u_1}{i^2} + \frac{\Delta^2 u_1}{i^3} + \ldots$$

which is identical with Mr. Gray's formula for the same problem, as given in a foot-note, page 93.

When the differences of the series of payments do not vanish after a given finite number of orders, the formulæ have to be used with caution; as if the number of payments be great, the error caused by the neglected differences may be considerable. Let us take the following case:

First payment	1·000000	+·010000	+·000100	+·000001	+·000000
Second "	1·010000	+·010100	+·000101	+ . .	
Third "	1·020100	+·010201	+ . .		
Fourth "	1·030301	+ . .			
. . .	. . .				

Now these four terms are in exact geometrical progression, the ratio being 1·01, and we see that the differences diminish with

considerable rapidity,—the fourth having no significant figure in the sixth decimal place. Let it be required to find the value of the perpetuity, at 5 per Cent., by means of the formula $\frac{u_1}{i}+\frac{\Delta u_1}{i^2}+\frac{\Delta^2 u_1}{i^3}+\frac{\Delta^3 u_1}{i^4}$.

$$
\begin{array}{r|l}
\Delta^3 u_1 = \cdot 000001 & \times 20 \ \left(=\frac{1}{i}\right) \\
\hline
\cdot 000020 & \\
\Delta^2 u_1 = \cdot 000100 & \\
\hline
\cdot 000120 & \times 20 \\
\hline
\cdot 002400 & \\
\Delta u_1 = \cdot 010000 & \\
\hline
\cdot 012400 & \times 20 \\
\hline
\cdot 248000 & \\
u_1 = 1 \cdot 000000 & \\
\hline
1 \cdot 248000 & \times 20 \\
\hline
24 \cdot 960000 &
\end{array}
$$

The true value of this perpetuity is easily found, thus: nth payment $=(1\cdot 01)^{n-1}$; present value of same $=(1\cdot 01)^{n-1}.(1\cdot 05)^{-n}$ $=\frac{1}{1\cdot 01}\cdot\left(\frac{1\cdot 05}{1\cdot 01}\right)^{-n}$;—the sum of which expression from 1 to infinity is $\frac{1}{1\cdot 01}\cdot\frac{1}{\frac{1\cdot 05}{1\cdot 01}-1}=\frac{1}{\cdot 04}=25$.* So that although we have used six decimal places in determining the differences, we do not get more than one correct decimal place in the result. This will not be wondered at when it is considered that the coefficient of the fourth difference $\left(\frac{1}{i^5}\right)$—which is not taken into account—amounts to no less than 3,200,000. Of course, however, any required degree of accuracy may be attained by computing a sufficient number of terms to a sufficient number of decimal places;—provided only that the differences of the series of payments diminishes faster than the increase in the coefficients.

* From this it is seen that the value of the increasing perpetuity is equal to the value of a fixed perpetuity of £1. calculated upon the assumption that the rate of interest is diminished by the rate of increase in the payments. Since the text was written I have received a communication from Mr. H. Mountcastle, of the Northern Assurance Office, pointing out the same result. The proposition is by no means *self-evident*, and it is not *true* for an annuity for a limited term.

I have stated that Baily's treatment of the subject of Increasing Annuities is not in my opinion satisfactory. Indeed when it is considered that his investigation is confined to Annuities whose successive payments are the several powers of the natural numbers, and that he makes no attempt to show that such Annuities are available for the solution of cases where the law of progression is different, it may naturally be asked why Baily should have devoted two chapters of his work to a subject invested with so limited a share of interest. The answer is found in an observation which he makes at the commencement of his investigation, from which it appears that the object was more especially to aid in the computation of life annuities on the principle of De Moivre's hypothesis; an object which the progress of the science of life contingencies has deprived of the importance which it possessed in Baily's eyes.

In Baily's arrangement the nth payment of an annuity of the tth order is n^{t-1},—while in that which I propose to adopt it is $\frac{(n-1)(n-2)\ldots(n-t+1)}{1.2\ldots.(t-1)}$. The advantage attending this substitution, consists in the fact that we are thereby put in possession of a key to the solution of all cases in which the differences of the payments vanish or become insignificant after a definite number of orders.

I think I have seen it pointed out, but cannot at present say where, that a life annuity may be considered as an annuity certain the successive payments of which are $p_{x|1}, p_{x|2}, p_{x|3}$, &c. Indeed the remark applies to any series the successive terms of which are discounted. It has therefore occurred to me—although I have made no attempt to verify the suggestion—that the preceding method might possibly be of use occasionally in the solution of certain problems in Life Contingencies.

The curious result pointed out by Mr. Makeham in the footnote on p. 198 may be thus illustrated. Let the successive annual payments of any benefit be $1+j$, $(1+j)^2$, $(1+j)^3$, ...; then following the course of reasoning used on p. 146, we see that the value of the benefit, estimated at a rate of interest i, is the same as that of a uniform benefit of £1, calculated at a rate of interest, I, equal to $\frac{i-j}{1+j}$. In the case of a perpetual annuity certain, the value will be $\frac{1}{\text{I}}$, or $\frac{1+j}{i-j}$; and is therefore equal to that of a perpetual annuity certain of $1+j$, calculated at a rate of interest $i-j$. Ed. *J. I. A.*